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Spin-wave expansion for the O(2) Heisenberg spin model in $(d+1)$ dimensions

C J Hamer† and Zheng Weihong‡

School of Physics, University of New South Wales, PO Box 1, Kensington, NSW 2033, Australia

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Abstract. Spin-wave expansions are carried out for the Hamiltonian version of the O(2) Heisenberg spin model, and evaluated for the cases of the (1+1)-dimensional linear chain, and the (2+1)-dimensional square and triangular lattices. Expressions are obtained for the bulk limits and finite-size corrections to the ground-state energy, the one-boson excitation energy, the magnetization and susceptibility at low 'temperatures'. Results are derived for the critical parameters in the (1+1)-dimensional model, and the Goldstone mode parameters in the (2+1)-dimensional models. Comparison is made with the finite-size scaling predictions of Fisher and Privman.

1. Introduction

The O(2) Heisenberg spin model provides a rich assortment of interesting physical phenomena. In two dimensions, Kosterlitz and Thouless (1973) used it as the first example of a new sort of phase transition, involving the unbinding of topological vortex-antivortex pairs. There is no long-range order at low temperatures, in accordance with the Mermin–Wagner theorem (Mermin and Wagner 1966), but the model displays a continuous line of critical behaviour there, characterized by a varying correlation length index η. The finite-size scaling behaviour of the model can tell us a great deal about the critical behaviour in this region, when the theory of conformal invariance is employed (Cardy 1987). A *finite-size scaling* approach was used by Luck (1982) to obtain a spin-wave expansion for the critical index η at low temperatures.

In three dimensions, the model is expected to undergo a standard second-order phase transition, and at low temperature the usual first-order magnetic transition line occurs. There the O(2) symmetry is spontaneously broken, resulting in the appearance of a massless Goldstone boson in the theory. This produces some new and interesting finite-size scaling phenomena, which were discussed briefly by Cardy and Nightingale (1983), and more extensively by Fisher and Privman (1985), assuming that the renormalization group behaviour is dominated by a discontinuity fixed point at zero temperature. Recently, the effect of the Goldstone modes on the finite-size scaling behaviour has been discussed more systematically by Hasenfratz and Leutwyler (1990), using chiral perturbation theory.

† E-mail: 87b0090@cumulus.csd.unsw.oz.au

‡ E-mail: zwh@newt.phys.unsw.edu.au

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In this paper we set out to study the low-temperature behaviour of the Hamiltonian version of the O(2) Heisenberg model, by means of a spin-wave expansion. The Hamiltonian version was formulated by Hamer *et al* (1979), and has since been studied by means of series expansions (Hamer and Richardson 1981, Hornby and Barber 1985), finite lattice calculations (Roomany and Wyld 1980, Hamer and Barber 1981, Allton and Hamer 1988), Monte Carlo methods (Heys and Stump 1984, Stump 1986), and some analytic works (Migdal 1975, Stump 1980, Mattis 1984). Here we study both the bulk behaviour and the finite-size scaling corrections for the ground-state energy, the one-boson excitation energy, the magnetization and the susceptibility, using the spin-wave expansion.

The results are in excellent agreement with theoretical expectations, in particular with the finite-size scaling predictions of Fisher and Privman (1985). The Hamiltonian model, in which one (time) dimension is infinite, corresponds to the 'cylindrical' geometry of Fisher and Privman. Whereas the finite-size scaling corrections at low temperature vary exponentially with lattice size M in a model with discrete symmetry, it turns out that they vary as integer powers of M in a model with continuous symmetry such as the present one. The Goldstone modes, which control the finite-size scaling behaviour, are characterized by three parameters: the helicity modulus, the spontaneous magnetization, and the 'speed of light'. Estimates are given for those quantities.

In section 2 of the paper the spin-wave expansion formalism is presented. The results for the bulk properties of the system are given in section 3 and 4. In section 5 the finite-size corrections are discussed, beginning with the 'zero mode' sector, which corresponds to the 'degeneracy kernel' treated by Fisher and Privman (1985). Our conclusions are summarized in section 6.

2. Spin-wave expansion

The quantum Hamiltonian for the O(2) Heisenberg spin model is (Hamer *et al* 1979)

$$H = \sum_i J^2(i) - x \sum_{\langle ij \rangle} \boldsymbol{n}(i) \cdot \boldsymbol{n}(j) - h \sum_i n_1(i) \tag{1}$$

where $\langle ij \rangle$ denotes nearest-neighbour pairs, and $\boldsymbol{n}(i)$ is a two-component spin vector at site i, normalized to unity, so that

$$\boldsymbol{n}(i) \equiv (n_1(i), n_2(i)) = (\cos\theta(i), \sin\theta(i)) \tag{2}$$

while $J(i)$ is the angular momentum operator conjugate to $\theta(i)$, which can take any integer eigenvalue. The commutation relations are

$$[J(i), \theta(j)] = -\mathrm{i}\delta_{i,j} \,. \tag{3}$$

The parameter x is the 'thermal' variable†, while h is the magnetic field. Let us set $h = 0$ for the time being, and rewrite the Hamiltonian in the equivalent form

$$H = \sum_i J^2(i) - x \sum_{\langle ij \rangle} \cos[\theta(i) - \theta(j)] \,. \tag{4}$$

† In the classical limit, kT is proportional to $x^{-1/2}$ (Hamer *et al* 1979).

The spin-wave expansion is a low temperature expansion, corresponding to large x value. At large x, the second term in equation (4) dominates, and for low-energy states the differences $[\theta(i) - \theta(j)]$ will all be small, so that one is led to make a power series expansion of the cosine. Thus

$$H = \sum_i J^2(i) - \tfrac{1}{2} z x N + \tfrac{1}{2} x \sum_{(ij)} \left(|\theta(i) - \theta(j)|^2 - \tfrac{1}{12} |\theta(i) - \theta(j)|^4 + \tfrac{1}{360} |\theta(i) - \theta(j)|^6 \right) + \mathrm{O}(\theta^8) \tag{5}$$

where N is the total number of lattice sites, and z the 'co-ordination number' (number of nearest neighbours for each site). Next, we perform a Fourier series expansion

$$\theta_k = \frac{1}{\sqrt{N}} \sum_m \theta(m) \exp(-\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{m}) \qquad J_k = \frac{1}{\sqrt{N}} \sum_m J(m) \exp(\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{m}). \tag{6}$$

Note that then

$$\theta_k^\dagger = \theta_{-k} \qquad J_k^\dagger = J_{-k} \tag{7}$$

and the commutation relations in momentum space are

$$[J_k, \theta_{k'}] = -\mathrm{i} \delta_{k,k'} . \tag{8}$$

Then the Hamiltonian becomes

$$\begin{aligned} H = {} & \sum_k J_k^\dagger J_k - \frac{xzN}{2} + \frac{xz}{2} \sum_k (1 - \gamma_k) \theta_k^\dagger \theta_k \\ & - \frac{xz}{24N} \sum_{k_1, \ldots, k_4} \delta_{1+2+3+4,0} \theta_1 \theta_2 \theta_3 \theta_4 R(k_1, k_2, k_3, k_4) \\ & + \frac{xz}{720N^2} \sum_{k_1, \ldots, k_6} \delta_{1+2+3+4+5+6,0} \theta_1 \theta_2 \theta_3 \theta_4 \theta_5 \theta_6 \\ & \times \Big(1 - \sum_i \gamma_i + \sum_{ij} \gamma_{i+j} - \sum_{i,j>1} \gamma_{1+i+j} \Big) + \mathrm{O}(\theta^8) \end{aligned} \tag{9}$$

where

$$\gamma_k = \frac{1}{z} \sum_\mu \exp(\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{\mu}) \tag{10}$$

is the 'structure factor', with $\boldsymbol{\mu}$ denoting the unit vectors connecting nearest neighbours on the lattice, R is defined by

$$R(k_1, k_2, k_3, k_4) = 1 - \sum_i \gamma_i + \gamma_{1+2} + \gamma_{1+3} + \gamma_{1+4} \tag{11}$$

and in the summations we adopt the convention of writing 1 instead of k_1, etc.

The quadratic terms in H can now be diagonalized by a Bogoliubov transformation

$$\alpha_k = (1/\sqrt{2})(\tan\phi_k J_k - \mathrm{i}\cot\phi_k \theta_k^\dagger) \tag{12}$$

where $\phi_k = \phi_{-k}$, and

$$\cot\phi_k = \left[\tfrac{1}{2}xz(1-\gamma_k)\right]^{1/4}. \tag{13}$$

Then one finds that $\alpha_k^\dagger$, α_k obey the commutation relations of Bose creation and destruction operators

$$[\alpha_k, \alpha_{k'}^\dagger] = \delta_{k,k'} \tag{14}$$

and after normal ordering the Hamiltonian becomes

$$\begin{aligned} H = N&\left[-\frac{xz}{2} + \left(\frac{xz}{2}\right)^{1/2} C_1 - \frac{C_1^2}{8} + \frac{C_1^3}{48}\left(\frac{2}{xz}\right)^{1/2}\right] + \left[\sqrt{2xz} - \frac{C_1}{2} + \frac{C_1^2}{8}\left(\frac{2}{xz}\right)^{1/2}\right] \\ &\times \sum_k n_k\sqrt{1-\gamma_k} + \sum_{k_1,k_2} \delta_{1+2,0} V_0^{(0)}(\alpha_{-1}\alpha_{-2} + \alpha_1^\dagger\alpha_2^\dagger) \\ &+ \sum_{k_1,\ldots,k_4} \delta_{1+2+3+4,0} V_1^{(0)}\Big[(\alpha_1^\dagger\alpha_2^\dagger\alpha_3^\dagger\alpha_4^\dagger + \alpha_{-1}\alpha_{-2}\alpha_{-3}\alpha_{-4}) \\ &- 4(\alpha_1^\dagger\alpha_{-2}\alpha_{-3}\alpha_{-4} + \alpha_2^\dagger\alpha_3^\dagger\alpha_4^\dagger\alpha_{-1}) + 6\alpha_1^\dagger\alpha_2^\dagger\alpha_{-3}\alpha_{-4}\Big] + \mathrm{O}(x^{-1}) \end{aligned} \tag{15}$$

where the lattice sums C_n are defined as

$$C_n = \frac{1}{N}\sum_k (1-\gamma_k)^{n/2} \tag{16}$$

$n_k = \alpha_k^\dagger\alpha_k$ is the boson number operator, and the two-particle vertex factor $V_0^{(0)}(k_1,k_2)$ and four-particle vertex factor $V_1^{(0)}(k_1,k_2,k_3,k_4)$ are

$$\begin{aligned} V_0^{(0)}(k_1,k_2) &= \frac{C_1\sqrt{2xz}}{16}(2-\gamma_1-\gamma_2)\tan\phi_1\tan\phi_2 \\ V_1^{(0)}(k_1,k_2,k_3,k_4) &= -\frac{xz}{96N} R(k_1,k_2,k_3,k_4)\prod_{i=1}^{4}\tan\phi_i. \end{aligned} \tag{17}$$

It can be seen that the spin-wave expansion is effectively an expansion in powers of $(xz)^{-1/2}$. It is expected to be an asymptotic expansion, and does not reproduce (for instance) the non-perturbative topological effects which are responsible for the Kosterlitz–Thouless transition in (1+1) dimensions.

3. Bulk properties—thermal

3.1. Ground-state energy

Using Rayleigh–Schrödinger perturbation theory, with the perturbation terms being those containing the vertex factors V_i, one can find the ground-state energy per site from the Hamiltonian (15)

$$E_0/N = -\frac{xz}{2}+\left(\frac{xz}{2}\right)^{1/2}C_1-\frac{C_1^2}{8}+\frac{C_1^3}{48}\left(\frac{2}{xz}\right)^{1/2}+\Delta E_a^{(-1/2)}+\Delta E_b^{(-1/2)}+\mathrm{O}(x^{-1}) \tag{18}$$

to third order in the spin-wave expansion, where

$$\Delta E_a^{(-1/2)} = -\frac{C_1^3}{32}\left(\frac{2}{xz}\right)^{1/2} \tag{19}$$

and

$$\Delta E_b^{(-1/2)} = -\frac{D_1}{192}\left(\frac{2}{xz}\right)^{1/2} \tag{20}$$

are the contributions from the two perturbation diagrams shows in figure 1. Here D_1 is defined by

$$D_1 = \frac{1}{N^3}\sum_{k_1,\ldots,k_4}\delta_{1+2+3+4,0}\frac{[R(k_1,k_2,k_3,k_4)]^2}{(\sum_{i=1}^4\sqrt{1-\gamma_i})(\prod_{i=1}^4\sqrt{1-\gamma_i})}\,. \tag{21}$$

Figure 1. The perturbation diagrams that contribute to the ground state energy E_0/N. The crosses represent the interaction vertices; the lines represent boson excitations in the intermediate states.

Using the values given for various lattice sums in the appendix, one finds the following results for specific lattices:

1. One-dimensional linear chain

$$E_0/N = -x + 2\sqrt{2x}/\pi - 1/\pi^2 - 0.01080x^{-1/2} + \mathrm{O}(x^{-1})\,. \tag{22}$$

2. Two-dimensional square lattice

$$E_0/N = -2x + 1.35495\sqrt{x} - 0.114742 - 0.008255x^{-1/2} + \mathrm{O}(x^{-1})\,. \tag{23}$$

3. Two-dimensional triangular lattice

$$E_0/N = -3x + 1.67618\sqrt{x} - 0.117066 - 0.00669x^{-1/2} + \mathrm{O}(x^{-1})\,. \tag{24}$$

3.2. Dispersion relation

The energy $E(k)$ of a single-boson state with momentum k can be derived from the Hamiltonian (15) as

$$E(k) = \sqrt{1-\gamma_k}\left[\sqrt{2xz} - \frac{C_1}{2} - \frac{1}{8}\left(\frac{2}{xz}\right)^{1/2}\left(\frac{C_1^2}{2} + \frac{D_3(k)}{3}\right) + \mathrm{O}(x^{-1})\right] \tag{25}$$

where

$$D_3(k) = \frac{1}{N^2}\sum_{k_2,\ldots,k_4} \delta_{k+2+3+4,0} \times \frac{[R(k,k_2,k_3,k_4)]^2 \sum_{i=2}^{4}(1-\gamma_i)^{1/2}}{(1-\gamma_k)[\prod_{i=2}^{4}(1-\gamma_i)^{1/2}][(\sum_{i=2}^{4}(1-\gamma_i)^{1/2})^2 - (1-\gamma_k)]}\,. \tag{26}$$

Besides diagonal terms, the five perturbation diagrams shown in figure 2 contribute to this result.

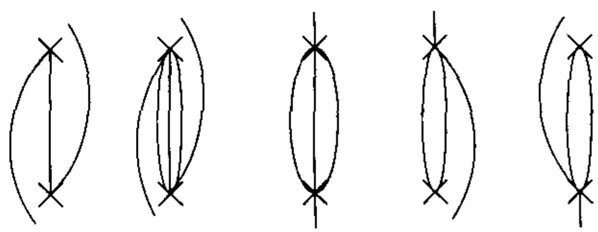

Figure 2. The perturbation diagrams that contribute to the energy $E(k)$ of a single-boson state.

Now at small k (for $(d+1)$ dimensions)

$$\sqrt{1-\gamma_k} \sim \frac{|k|}{\sqrt{2d}} \qquad \text{as } k \to 0 \tag{27}$$

while

$$D_3(k) \sim \text{constant} \qquad \text{as } k \to 0 \tag{28}$$

and so one finds a linear dispersion relation at low momentum

$$E(k) \sim v(x)|k| \qquad \text{as } k \to 0 \tag{29}$$

corresponding to a massless boson field. The factor $v(x)$ is the spin-wave velocity or 'speed of light' in the model, which is then

$$v(x) = \frac{1}{\sqrt{2d}}\left[\sqrt{2xz} - \frac{C_1}{2} - \frac{1}{8}\left(\frac{2}{xz}\right)^{1/2}\left(\frac{C_1^2}{2} + \frac{D_3(0)}{3}\right) + \mathrm{O}(x^{-1})\right]. \tag{30}$$

In $(1+1)$ dimensions, the presence of this massless field is an indicator of the criticality of the model at low temperatures. In $(2+1)$ dimensions, it corresponds to the Goldstone mode produced by spontaneous breakdown of the continuous O(2) symmetry.

For specific lattices, one finds the following results:

1. One-dimensional linear chain

$$v(x) = \sqrt{2x} - \frac{1}{\pi} - 0.06126111x^{-1/2} + O(x^{-1}) . \tag{31}$$

2. Two-dimensional square lattice

$$v(x) = \sqrt{2x} - 0.239523 - 0.0278391x^{-1/2} + O(x^{-1}) . \tag{32}$$

3. Two-dimensional triangular lattice

$$v(x) = \sqrt{3x} - 0.241936 - 0.0214988x^{-1/2} + O(x^{-1}) . \tag{33}$$

4. Bulk properties—magnetic

The magnetic operator in the Hamiltonian (1) is

$$V = \sum_i n_1(i) = \sum_i \cos\theta(i) . \tag{34}$$

If we expand the cosine in a power series†, and then follow the same procedure as in section 2 with the magnetic field h non-zero, then everything follows as before, where now

$$\cot\phi_k = \left[\frac{xz}{2}(1-\gamma_k) + \frac{h}{2}\right]^{1/4} . \tag{35}$$

The final, normal-ordered Hamiltonian is

$$\begin{aligned} H = N\Bigg[&-\frac{xz}{2} - h + \frac{1}{N}\sum_k \left[\frac{xz}{2}(1-\gamma_k) + \frac{h}{2}\right]^{1/2} \\ &- \frac{1}{16N^2}\sum_{k_1,k_2} \frac{2xz(1-\gamma_1)(1-\gamma_2) + h}{\sqrt{xz(1-\gamma_1)+h}\sqrt{xz(1-\gamma_2)+h}} \\ &+ \frac{\sqrt{2}}{192N^3}\sum_{k_1,k_2,k_3} \frac{4xz(1-\gamma_1)(1-\gamma_2)(1-\gamma_3)+h}{\prod_{i=1}^3 [xz(1-\gamma_i)+h]^{1/2}}\Bigg] \\ &+ \sum_k \Bigg[\sqrt{2xz(1-\gamma_k)+2h} \\ &- \frac{1}{4N}\sum_{k_1} \frac{2xz(1-\gamma_k)(1-\gamma_1)+h}{\sqrt{xz(1-\gamma_1)+h}\sqrt{xz(1-\gamma_k)+h}}\Bigg] n_k + \end{aligned}$$

† This procedure will be reconsidered more carefully in section 5.

$$+\sum_k \mathcal{V}_0^{(0)}(\alpha_k\alpha_{-k}+\alpha_k^\dagger\alpha_{-k}^\dagger)$$
$$+\sum_{k_1,\ldots,k_4}\delta_{1+2+3+4,0}\mathcal{V}_1^{(0)}\left[(\alpha_1^\dagger\alpha_2^\dagger\alpha_3^\dagger\alpha_4^\dagger+\alpha_{-1}\alpha_{-2}\alpha_{-3}\alpha_{-4})\right.$$
$$\left.-4(\alpha_1^\dagger\alpha_{-2}\alpha_{-3}\alpha_{-4}+\alpha_2^\dagger\alpha_3^\dagger\alpha_4^\dagger\alpha_{-1})+6\alpha_1^\dagger\alpha_2^\dagger\alpha_{-3}\alpha_{-4}\right]+\mathrm{O}(x^{-1}) \tag{36}$$

where

$$\mathcal{V}_0^{(0)}(k)=\frac{1}{8N}\sum_{k_1}\frac{2xz(1-\gamma_1)(1-\gamma_k)+h}{\sqrt{xz(1-\gamma_1)+h}\sqrt{xz(1-\gamma_k)+h}}$$
$$\mathcal{V}_1^{(0)}(k_1,k_2,k_3,k_4)=-\frac{1}{48N}\frac{xzR(k_1,k_2,k_3,k_4)+h}{\prod_{i=1}^4[xz(1-\gamma_i)+h]^{1/4}}. \tag{37}$$

The ground-state energy per site is then

$$\frac{E(h)}{N}=-\frac{xz}{2}-h+\frac{1}{N}\sum_k\left[\frac{xz}{2}(1-\gamma_k)+\frac{h}{2}\right]^{1/2}$$
$$-\frac{1}{16N^2}\sum_{k_1,k_2}\frac{2xz(1-\gamma_1)(1-\gamma_2)+h}{\sqrt{xz(1-\gamma_1)+h}\sqrt{xz(1-\gamma_2)+h}}$$
$$+\frac{\sqrt{2}}{192N^3}\sum_{k_1,k_2,k_3}\frac{4xz(1-\gamma_1)(1-\gamma_2)(1-\gamma_3)+h}{\prod_{i=1}^3[xz(1-\gamma_i)+h]^{1/2}}$$
$$+\Delta\mathcal{E}_a^{(-1/2)}+\Delta\mathcal{E}_b^{(-1/2)}+\mathrm{O}(x^{-1}) \tag{38}$$

where

$$\Delta\mathcal{E}_a^{(-1/2)}=-\frac{1}{64N^3\sqrt{2}}\times\sum_{k,k_1,k_2}\frac{[2xz(1-\gamma_k)(1-\gamma_1)+h][2xz(1-\gamma_k)(1-\gamma_2)+h]}{[xz(1-\gamma_1)+h]^{1/2}[xz(1-\gamma_2)+h]^{1/2}[xz(1-\gamma_k)+h]^{3/2}} \tag{39}$$

$$\Delta\mathcal{E}_b^{(-\frac{1}{2})}=-\frac{1}{192N^3}\left(\frac{2}{xz}\right)^{1/2}\sum_{k_1,\ldots,k_4}\delta_{1+2+3+4,0}$$
$$\times\frac{[R(k_1,k_2,k_3,k_4)+h/xz]^2}{\prod_{i=1}^4[(1-\gamma_i)+h/xz]^{1/2}\sum_{i=1}^4[(1-\gamma_i)+h/xz]^{1/2}}.$$

4.1. Spontaneous magnetization

The spontaneous magnetization, defined by

$$\Sigma=-\frac{1}{N}\frac{\partial E_0}{\partial h}\bigg|_{h=0} \tag{40}$$

is computed to be

$$\Sigma = 1-\frac{C_{-1}}{2\sqrt{2xz}}-\frac{C_{-1}}{16xz}(2C_1-C_{-1})-\frac{1}{48(2xz)^{3/2}}[C_{-1}(C_{-1}^2-6C_1C_{-1}+9C_1^2)-D_2] \tag{41}$$

where

$$D_2 = \frac{1}{N^3}\sum_{k_1,\ldots,k_4}\delta_{1+2+3+4,0}\frac{\left[2-\frac{R}{2}\frac{\sum_i(1-\gamma_i)^{-1/2}}{\sum_i(1-\gamma_i)^{1/2}}+\frac{R}{2}\sum_i(1-\gamma_i)^{-1}\right]R}{[\prod_{i=1}^4(1-\gamma_i)^{1/2}][\sum_{i=1}^4(1-\gamma_i)^{1/2}]}. \tag{42}$$

1. One-dimensional linear chain. In this case the lattice constant C_{-1} diverges, and so the spin-wave expansion does not converge. This may be an indication, however, of the expected behaviour: at zero temperature ($x \to \infty$) $\Sigma = 1$, but at any finite temperature the spontaneous magnetization is strictly zero, according to the Mermin–Wagner (1966) theorem.
2. Two-dimensional square lattice

$$\Sigma = 1 - 0.227293x^{-1/2} - 0.012665x^{-1} - 0.001675x^{-3/2} + O(x^{-2}). \tag{43}$$

3. Two-dimensional triangular lattice

$$\Sigma = 1 - 0.178708x^{-1/2} - 0.008994x^{-1} - 0.001107x^{-3/2} + O(x^{-2}). \tag{44}$$

4.2. Susceptibility

The magnetic susceptibility is

$$\chi = -\frac{1}{N}\frac{\partial^2 E_0}{\partial h^2}. \tag{45}$$

This quantity diverges at $h = 0$ in both (1+1) and (2+1) dimensions. To study how it diverges, we first look at the two order term of χ at small but finite h

$$\chi \sim \frac{\mathcal{C}_{-3}}{4\sqrt{2}}+\frac{\mathcal{C}_{-3}(3\mathcal{C}_1-2xz\mathcal{C}_{-1})}{16xz}+\frac{h(xz\mathcal{C}_{-3}^2+3xz\mathcal{C}_{-1}\mathcal{C}_{-5}-6\mathcal{C}_{-5}\mathcal{C}_1)}{32xz} \tag{46}$$

where

$$\mathcal{C}_n = \frac{1}{N}\sum_k[xz(1-\gamma_k)+h]^{n/2} \tag{47}$$

Replacing the sum by an integral in the bulk limit, and using the asymptotic expansion near $h = 0$ by Hamer *et al* (1991) and Zheng *et al* (1991), we can get χ at leading order of h:

1. One-dimensional linear chain

$$\chi \sim \frac{1}{4h\pi\sqrt{2x}}+\frac{5+\ln(h/x)}{32h\pi^2x}+O(x^{-3/2}). \tag{48}$$

Again, we see that the spin-wave expansion does not converge for the magnetic derivatives as $h \to 0$.

2. Two-dimensional lattices

$$\chi \sim \begin{cases} \dfrac{h^{-1/2}}{8\sqrt{2}\pi x} - \dfrac{0.0001947h^{-1/2}}{\pi x^{3/2}} + O(x^{-2}) & \text{(square lattice)} \\[2ex] \dfrac{h^{-1/2}}{8\sqrt{6}\pi x} + \dfrac{0.0001836h^{-1/2}}{x^{3/2}} + O(x^{-2}) & \text{(triangular lattice).} \end{cases} \tag{49}$$

This behaviour matches that of the three-dimensional Euclidean model derived by Vaks *et al* (1967). Equation (49) agrees with the prediction of Fisher and Privman (1985) that χ should diverge as $|h|^{(d-4)/2}$ as $h \to 0$.

5. Finite-lattice corrections

The finite-size scaling corrections can give us a great deal of information about the model (Barber 1983), especially when the theory of conformal invariance is applied at criticality in (1+1) dimensions (Cardy 1987). The spin-wave 'zero mode', which we have neglected hitherto, plays a crucial role in the finite-lattice corrections, so we begin with a more careful consideration of its effects.

5.1. The zero mode

Consider first the case with magnetic field $h = 0$. If we separate out the $\boldsymbol{k} = 0$ terms from the Hamiltonian (9), we find

$$H_{k=0} = J_0^\dagger J_0 \tag{50}$$

which is already diagonal. The thermal operator involves only differences $[\theta(i) - \theta(j)]$, and therefore has no dependence on the overall 'average' orientation θ_0 of the system, at any order. The eigenvalues of J_0 are easily seen to be

$$J_0^\dagger = J_0 = \frac{l}{\sqrt{N}} \qquad (l = \text{integer}) \tag{51}$$

and thus the zero-mode energy eigenvalues are

$$E_{k=0} = \frac{l^2}{N} \qquad (l = \text{integer}) \tag{52}$$

exact to all orders in the spin-wave expansion, with corresponding eigenvectors

$$|\Psi^l_{k=0}(\theta_0)\rangle = \frac{1}{\sqrt{2\pi}N^{1/4}} \exp\left[\frac{\mathrm{i}l}{\sqrt{N}}\theta_0\right] . \tag{53}$$

Next, consider the magnetic operator

$$V = \sum_m \cos\theta(m) . \tag{54}$$

As $h \to 0$, this magnetic term represents a small perturbation, and it is not strictly correct to assume all the angles $\theta(m)$ are small, and to expand the cosines in a power series. The zero mode must be separated out first. Define

$$\theta(m) = \frac{\theta_0}{\sqrt{N}} + \theta^-(m) \tag{55}$$

then

$$\begin{aligned}\cos\theta(m) &= \cos\left(\frac{\theta_0}{\sqrt{N}}\right)\cos\theta^-(m) - \sin\left(\frac{\theta_0}{\sqrt{N}}\right)\sin\theta^-(m) \\ &\simeq \cos\left(\frac{\theta_0}{\sqrt{N}}\right)\left[1 - \frac{1}{2}(\theta^-(m))^2\right] - \theta^-(m)\sin\left(\frac{\theta_0}{\sqrt{N}}\right)\end{aligned} \tag{56}$$

since $\theta^-(m)$ can be expressed purely in terms of the differences $[\theta(i) - \theta(j)]$. Summing over sites m, one finds after a little algebra

$$V = \sum_m \cos\theta(m) = \cos\left(\frac{\theta_0}{\sqrt{N}}\right)\left[N - \frac{1}{2}\sum_{k\neq 0}\theta_k^\dagger\theta_k\right] + O(\theta^4)\,. \tag{57}$$

With these preliminaries, we are equipped to discuss the leading finite-size corrections.

5.2. Ground-state energy

The zero mode plays no special role here, and the finite-size corrections arise simply from the difference between the finite-lattice values of the lattice constants C_n and D_n and their bulk values, as listed in the appendix. Thus one finds the following results:

1. One-dimensional linear chain

$$E_0/N \sim e_\infty - (\pi/6M^2)(\sqrt{2x} - 1/\pi - 0.06126111x^{-1/2} + O(x^{-1})) \qquad \text{as } M \to \infty \tag{58}$$

where $M = N$ is the lattice size, and e_∞ is the bulk value. Now according to the theory of conformal invariance (Blöte *et al* 1986, Affleck 1986), the leading finite-size correction at a critical point is

$$E_0/N \sim e_\infty - \pi v(x)c/6M^2 \qquad \text{as } M \to \infty \tag{59}$$

where $v(x)$ is a scale factor equal to the spin-wave velocity, and c is the conformal anomaly, which characterizes the universality class of the critical point, and the allowed set of critical exponents. Comparing equation (58) with (59), and recalling equation (31) for the spin-wave velocity $v(x)$, we see that the conformal anomaly is

$$c = 1 \tag{60}$$

precisely, through third-order in the spin-wave expansion. This is expected to be an exact result for this model; it allows the possibility of continuously varying critical exponents.

2. Two-dimensional square lattices

$$E_0/N \sim e_\infty - (1/M^3)(1.02x^{1/2} - 0.172 + O(x^{-1/2})) \qquad \text{as } M \to \infty . \tag{61}$$

3. Two-dimensional triangular lattices

$$E_0/N \sim e_\infty - (1/M^3)(1.351x^{1/2} - 0.189 + O(x^{-1/2})) \qquad \text{as } M \to \infty . \tag{62}$$

The $1/M^3$ dependence of the finite lattice corrections in (2+1) dimensions was predicted by Fisher and Privman (1985), and is the same as that expected at a second-order critical point. The amplitude of the corrections has no special significance, so far as we are aware.

5.3. Mass gap

The mass gap in the model is just the gap between the lowest energy states in the zero mode sector, which from equation (52) is simply

$$F_N = 1/N \tag{63}$$

exact to all orders in the spin-wave expansion. The only corrections will be non-perturbative 'tunnelling' effects, of order $\exp(-\text{constant} \times \sqrt{x})$ (Hamer and Barber 1981)

1. One-dimensional linear chain

$$F_N = 1/M . \tag{64}$$

A $1/M$ dependence for the mass gap is the primary signal of a critical point in finite-size scaling theory (Barber 1983). According the the theory of conformal invariance (Cardy 1984), the finite-size scaling amplitude is related to the correlation length exponent $\eta(x)$ by

$$F_M \sim \pi v(x)\eta(x)/M \qquad \text{as } M \to \infty \tag{65}$$

where again the scale factor $v(x)$ equals the spin-wave velocity for the Hamiltonian model. Comparing equations (64) and (65), we see that in this model

$$\eta(x) = (1/\pi v(x)) \tag{66}$$

i.e.

$$\eta(x) = \frac{1}{\pi\sqrt{2x}} + \frac{1}{2\pi^2 x} + \frac{0.02115265}{x^{3/2}} + O(x^{-2}) . \tag{67}$$

2. Two-dimensional square lattices

$$F_N = 1/M^2 . \tag{68}$$

The $1/M^2$ dependence is just that predicted by Fisher and Privman (1985). They show that the longitudinal correlation length $\xi_\parallel$ in the cylindrical geometry is given in terms of a quantity called the 'helicity modulus' Υ by

$$\xi_\parallel \sim \frac{2\Upsilon A}{kT} \qquad \text{as } M \to \infty \tag{69}$$

where A is the transverse cross-sectional area. If we invert the correlation length to get a mass gap, multiply by the ubiquitous scaling factor $v(x)$, replace kT by $\sqrt{2/x}$ (Hamer *et al* 1979), and set $A = M^2$ (lattice spacing $a = 1$), then the equivalent expression for the Hamiltonian model is

$$F_N \sim \frac{v(x)}{\sqrt{2x}\Upsilon(x)M^2} \qquad \text{as } M \to \infty . \tag{70}$$

Comparing (70) with (69), we see that the helicity modulus is

$$\Upsilon(x) = \frac{v(x)}{\sqrt{2x}} \tag{71}$$

in the spin-wave expansion, i.e.

$$\Upsilon(x) = 1 - 0.169368x^{-1/2} - 0.0196852x^{-1} + \mathrm{O}(x^{-3/2}) . \tag{72}$$

3. Two-dimensional triangular lattices, here again

$$F_N = 1/M^2 . \tag{73}$$

The argument is the same as for the square lattice, except in this case kT should be replaced by $1/\sqrt{x}$, and $A = \sqrt{3}M^2/2$; and hence

$$\Upsilon(x) = \frac{v(x)}{\sqrt{3x}} \tag{74}$$

i.e.

$$\Upsilon(x) = 1 - 0.139682x^{-1/2} - 0.0124123x^{-1} + \mathrm{O}(x^{-3/2}) . \tag{75}$$

5.4. Magnetization

The spontaneous magnetization is

$$\Sigma_N = -\frac{1}{N}\frac{\partial E_0}{\partial h}\bigg|_{h=0} = \frac{1}{N}\langle 0|V|0\rangle . \tag{76}$$

Now the zero mode sector is disjoint from the rest of the Hamiltonian, and therefore the matrix element in (76) is proportional to the zero-mode matrix element

$$\langle\Psi^0_{k=0}(\theta_0)|\cos\left(\frac{\theta_0}{\sqrt{N}}\right)|\Psi^0_{k=0}(\theta_0)\rangle = \langle\Psi^0_{k=0}(\theta_0)|\frac{L^+ + L^-}{2}|\Psi^0_{k=0}(\theta_0)\rangle = 0 \tag{77}$$

where $L^{\pm}$ are raising and lowering operators for the 'helicity' l in the zero-mode sector. Therefore

$$\Sigma_N = 0 \tag{78}$$

i.e. the spontaneous magnetization is zero on any finite lattice, in agreement with general theorems. Spontaneous symmetry breaking can only develop in the bulk limit.

5.5. Susceptibility

The susceptibility is

$$\chi_N = -\frac{1}{N}\frac{\partial^2 E_0}{\partial h^2}\bigg|_{h=0} = -\frac{2}{N}\sum_n \frac{|\langle 0|V|n\rangle|^2}{E_0^0 - E_n^0} \tag{79}$$

where E_n^0 denotes the energy eigenvalues when $h = 0$. The magnetic operator can be expanded in terms of boson operators as follows

$$\begin{aligned} V &= \cos\left(\frac{\theta_0}{\sqrt{N}}\right)\left[N - \frac{1}{2}\sum_{k\neq 0}\theta_k^\dagger\theta_k\right] + O(\theta^4) \\ &= \left(\frac{L^+ + L^-}{2}\right)\left[N - \frac{1}{2}\sum_{k\neq 0}\tan^2\phi_k\left(n_k + \frac{1}{2} - \frac{1}{2}(\alpha_k\alpha_{-k} + \alpha_k^\dagger\alpha_{-k}^\dagger)\right)\right] \\ &\quad + (\text{higher-order terms})\,. \end{aligned} \tag{80}$$

Therefore

$$\sum_n \frac{|\langle 0|V|n\rangle|^2}{E_0^0 - E_n^0} = \frac{1}{2}\left\{\frac{(N - \frac{1}{4}\sum_{k\neq 0}\tan^2\phi_k)^2}{(-1/N)} + \frac{2\sum_{k\neq 0}(\frac{1}{4}\tan^2\phi_k)^2}{(-1/N - 2\sqrt{2xz(1-\gamma_k)})}\right\} \tag{81}$$

in leading order, where the first term involves an excitation in the zero-mode sector only, while the second term involves a pair of excited bosons with momenta k and $-k$ as well. Hence one finds

$$\chi_N = N^2\left(1 - \frac{C_{-1}}{2\sqrt{2xz}}\right)^2 + \frac{C_{-3}}{8\sqrt{2}(xz)^{3/2}} + (\text{higher-order terms})\,. \tag{82}$$

1. One-dimensional linear chain. In this case, both C_{-1} and C_{-3} diverge as $M \to \infty$ (see appendix). The first term in (82) gives

$$\chi_N \simeq M^2\left(1 - \frac{\ln M}{2\pi\sqrt{2x}}\right) \simeq M^{2-1/\pi\sqrt{2x}} \simeq M^{2-\eta(x)} \tag{83}$$

to leading order, as predicted by finite-size scaling theory. Note, however, that these approximations are only valid for $\sqrt{x} \gg \ln M$.

2. Two-dimensional square lattice. Here only C_{-3} diverges as $M \to \infty$, and one finds

$$\chi_N \simeq M^4\left(1 - \frac{0.2273}{\sqrt{x}}\right)^2 + \frac{0.0032M}{x^{3/2}}\,. \tag{84}$$

The M^4 dependence of the leading term is that predicted by Cardy and Nightingale (1983) and Fisher and Privman (1985). Fisher and Privman predicted that the coefficient should be proportional to the square of the bulk spontaneous magnetization Σ, and indeed we see that

$$\chi_N \simeq M^4\Sigma^2 \tag{85}$$

to this order. This term develops into a delta function singularity in the bulk limit, corresponding to a finite discontinuity in the magnetization—that is, to the spontaneous magnetization.

The second term in (84) is referred to as the 'spin-wave contribution' by Fisher and Privman (1985). It is this term which gives rise to the $h^{-1/2}$ divergence in the bulk susceptibility as $h \to 0$. Note that the proportionality to M is consistent with finite-size scaling in the vicinity of the discontinuity fixed point: the mass gap is expected to vanish like h as $h \to 0$ in the bulk, whereas it vanishes like M^{-2} as $M \to \infty$ at $h \to 0$; correspondingly, the spin-wave contribution to the susceptibility diverges like $h^{-1/2}$ as $h \to 0$ in the bulk, while it diverges like M on the finite lattice.

3. Two-dimensional triangular lattice

$$\chi_N \simeq M^4\left(1 - \frac{0.1787}{\sqrt{x}}\right)^2 + \frac{0.00143M}{x^{3/2}} . \tag{86}$$

Similar comments apply.

6. Summary and conclusions

We have derived spin-wave expansions for the ground-state energy, one-boson excitation energy, magnetization and susceptibility of the O(2) Heisenberg spin model in $(d+1)$ dimensions. Both the bulk behaviour and the finite-size correlations have been calculated, for the cases of the (1+1)-dimensional linear chain, and the (2+1)-dimensional square and triangular lattices. The spin-wave expansion is expected to be an asymptotic expansion, valid at low temperatures or large couplings x. The results agree entirely with earlier theoretical predictions.

For a (1+1)-dimensional linear chain of M sites, the mass gap equals $1/M$ exactly in the spin-wave expansion (Hamer and Barber 1981), indicating the line of critical behaviour at low temperatures predicted by Kosterlitz and Thouless (1973) and Kosterlitz (1974). The conformal anomaly was found to be

$$c = 1 \tag{87}$$

precisely, through third order in the spin-wave expansion; this is expected to be an exact result. The correlation length index was found to be

$$\eta(x) = \frac{1}{\pi\sqrt{2x}} + \frac{1}{2\pi^2 x} + \frac{0.02115265}{x^{3/2}} + O(x^{-2}) \tag{88}$$

to be compared with Luck's (1982) result for the Euclidean version of the model

$$\eta(x) = \frac{T}{2\pi[1 - f(T)]} \tag{89}$$

where

$$f(T) = \tfrac{1}{4}T + \tfrac{5}{96}T^2 + 0.034127T^3 + O(T^4) . \tag{90}$$

There appears to be no very simple relationship between the Hamiltonian and Euclidean results.

The spin-wave velocity or 'speed of light' for the one-dimensional chain is related to $\eta(x)$ by

$$v(x) = 1/\pi\eta(x). \tag{91}$$

For the (2+1)-dimensional models at low temperatures, which are expected to exhibit a first-order magnetic transition, the finite-size scaling behaviour is just that predicted by Cardy and Nightingale (1983) and Fisher and Privman (1985). The mass gap is exactly $1/M^2$, neglecting non-perturbative effects, and the leading term in the susceptibility at zero magnetic field scales like M^4, for a lattice with M sites on a side. These effects are controlled by the 'zero mode' in the model, which plays a role equivalent to the 'degeneracy kernel' discussed by Fisher and Privman (1985).

In the bulk system, the O(2) symmetry is spontaneously broken, and the spin-wave excitations play the role of a massless Goldstone boson field. According to Fisher and Privman (1985) or Hasenfratz and Leutwyler (1990), the Goldstone modes and the finite-size scaling behaviour at leading order are governed by two parameters, the helicity modulus Υ and the spontaneous magnetization Σ, to which one must add the scale factor or spin-wave velocity v, in the Hamiltonian version of the model. In the spin-wave expansion, we find

$$v(x) = \begin{cases} \sqrt{2x} - 0.239523 - 0.0278391x^{-1/2} + O(x^{-1}) & \text{(square lattice)} \\ \sqrt{3x} - 0.241936 - 0.0214988x^{-1/2} + O(x^{-1}) & \text{(triangular lattice)} \end{cases}$$

$$\Sigma = \begin{cases} 1 - 0.227293x^{-1/2} - 0.012665x^{-1} - 0.001675x^{-3/2} + O(x^{-2}) & \text{(square lattice)} \\ 1 - 0.178708x^{-1/2} - 0.008994x^{-1} - 0.001107x^{-3/2} + O(x^{-2}) & \text{(triangular lattice)} \end{cases} \tag{92}$$

and

$$\Upsilon(x) = v(x)/\sqrt{\tfrac{1}{2}zx}. \tag{93}$$

with the particular normalization we have adopted for our Hamiltonian.

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Appendix. Calculation of C_n and D_n

Here we show how to calculate the bulk limits of lattice constants C_n and D_n ($n = 1, 2, 3$), together with their finite-lattice corrections.

The evaluation of C_n involves a summation over momentum k in the first Brillouin zone. For the bulk system, the momentum k is continuous over the first Brillouin zone, but for the finite-lattice system, the momentum k is discrete. For the following lattices, the structure factor γ_k, the first Brillouin zone for a bulk system and the discrete momentum k for a finite-lattice system are:

1. One-dimensional linear chain

$$\begin{aligned}
&\gamma_k = \cos(k_x a) \\
&0 < k_x a < 2\pi \qquad \text{(momentum } k\text{; bulk system)} \\
&k_x(i) = \frac{2\pi i}{Ma} \qquad i = 1, \dots, M \quad \text{(finite lattice system)}.
\end{aligned} \tag{A1}$$

2. Two-dimensional square lattice

$$\begin{aligned}
&\gamma_k = \frac{1}{2}[\cos(k_x a) + \cos(k_y a)] \\
&0 < k_x a,\ k_y a < 2\pi \qquad \text{(momentum } k\text{; bulk system)} \\
&k_x(i) = 2\pi i/Ma \qquad k_y(i) = 2\pi i/Ma \qquad i = 1, \dots, M \quad \text{(finite lattice system)}.
\end{aligned} \tag{A2}$$

3. Two-dimensional triangular lattice

$$\begin{aligned}
&\gamma_k = \tfrac{1}{3}[\cos(k_x a) + 2\cos(k_x a/2)\cos(\sqrt{3}k_y a/2)] \\
&0 < k_x a < 2\pi \qquad 0 < \sqrt{3}k_y a/2 < 2\pi \qquad \text{(momentum } k\text{; bulk system)} \\
&k_x(i) = \frac{2\pi i}{Ma} \qquad \frac{\sqrt{3}k_y(i)}{2} = \frac{2\pi i}{Ma} \qquad i = 1, \dots, M \quad \text{(finite lattice system)}.
\end{aligned} \tag{A3}$$

For the one-dimensional linear chain, C_n for the bulk system can easily be calculated exactly, and the finite-lattice correction to C_1 can also be carried out exactly by using the Euler–Maclaurin formula (Atkinson 1978). For the (2+1)-dimensional bulk system, the asymptotic expansions for $C_n(\infty)$ have been carried out by Hamer *et al* (1991) and Zheng *et al* (1991), and the finite lattice corrections can be evaluated by calculating $[C_n(M) - C_n(\infty)]M^{n+2}$ for a large lattice system, or by a least-square fit of $C_n(M)$ to the form $C_n(\infty) + A/M^{n+2} + B/M^{n+3} + C/M^{n+4}$. The constants $D_3(0)$ and D_1 in the case of (1+1) dimensions can also be calculated by a similar method. For (2+1) dimensions, instead of calculating D_1 and D_2 directly, we introduce a parameter t into D_1 and D_2

$$\begin{aligned}
D_1(t) &= \frac{1}{N^3}\sum_{k_1,\dots,k_4} \delta_{1+2+3+4,0} \frac{R^2}{(\sum_{i=1}^4 \sqrt{1-t\gamma_i})(\prod_{i=1}^4 \sqrt{1-t\gamma_i})} \\
D_2(t) &= \frac{1}{N^3}\sum_{k_1,\dots,k_4} \delta_{1+2+3+4,0} \frac{\left[2 - \frac{R}{2}\frac{\sum_i(1-t\gamma_i)^{-1/2}}{\sum_i(1-t\gamma_i)^{1/2}} + \frac{R}{2}\sum_i(1-t\gamma_i)^{-1}\right]R}{[\prod_{i=1}^4(1-t\gamma_i)^{1/2}][\sum_{i=1}^4(1-t\gamma_i)^{1/2}]}
\end{aligned} \tag{A4}$$

and expand $D_1(t)$ and $D_2(t)$ into a series in t using Mathematica, where the expansion coefficients can be integrated analytically. The results of the series expansion are:

Square lattice:

$$
\begin{aligned}
D_1(t) = {} & 11/16 - 35t/64 + 2299t^2/4096 - 0.4818115234375t^3 \\
& + 0.454561471939086914t^4 - 0.403501808643341064t^5 \\
& + 0.376768570626154542t^6 - 0.341403306287247688t^7 \\
& + 0.319433645036042435t^8 - 0.293592509247446287t^9 \\
& + 0.275949607092254467t^{10} - 0.256297599965265022t^{11} \\
& + 0.242045606162199269t^{12} - 0.226631423902937783t^{13} \\
& + 0.214972274242537484t^{14} - 0.202581371835823069t^{15} \\
& + 0.192912754853624253t^{16} - 0.182750979003280880t^{17} \\
& + 0.174629123714461590t^{18} + O(t^{19}) \\
D_2(t) = {} & 49/32 - 187t/128 + 15301t^2/8192 - 1.8829345703125t^3 \\
& + 2.08595311641693115t^4 - 2.10704937577247620t^5 \\
& + 2.23053067817818373t^6 - 2.24698880102369003t^7 \\
& + 2.32942795897724864t^8 - 2.34061539061690382t^9 \\
& + 2.39871912041739854t^{10} - 2.40558729524258563t^{11} \\
& + 2.44800332614562678t^{12} - 2.45157260139349534t^{13} \\
& + 2.48328626839828260t^{14} - 2.48438209682094625t^{15} + O(t^{16}) \,.
\end{aligned}
\tag{A5}
$$

Triangular lattice:

$$
\begin{aligned}
D_1(t) = {} & 13/24 - 5t/18 + 271t^2/1536 - 0.0737545814043209877t^3 \\
& + 0.0436201998235757746t^4 - 0.0154223186979568547t^5 \\
& + 0.0107438295984246t^6 - 0.0025021345481660t^7 \\
& + 0.0030214321737331 97t^8 + 0.0000082792123554 0t^9 \\
& + 0.00110337270030009t^{10} + 0.000368563937665 63t^{11} \\
& + 0.000545972191848 27t^{12} + 0.000336645824745 22t^{13} \\
& + 0.000337285526327 52t^{14} + 0.000260109518762 64t^{15} \\
& + 0.000235125860082 91t^{16} + O(t^{17}) \\
D_2(t) = {} & 21/16 - 19t/24 + 643t^2/1024 - 0.295563874421296296t^3 \\
& + 0.2117447558744454t^4 - 0.078547403155040t^5 \\
& + 0.067948332437108t^6 - 0.0139158838004841t^7 \\
& + 0.024470540206218t^8 + 0.002042745430904t^9 \\
& + 0.01128899453t^{10} + 0.0049446453t^{11} \\
& + 0.0068097607t^{12} + 0.00479251778925t^{13} \\
& + 0.00491528288384t^{14} + 0.004111568123838t^{15} + O(t^{16}) \,.
\end{aligned}
\tag{A6}
$$

Extrapolating the above series using integrated Dlog Padé approximants (Guttmann 1989) in $\delta = 1-(1-t)^{1/2}$ (Hamer *et al* 1991 and Zheng *et al* 1991), one can obtain an estimate at the limit $t = 1$.

The final results for C_n and D_n are summarized as:

1. One-dimensional linear chain

$$C_1 = \frac{2\sqrt{2}}{\pi} - \frac{\sqrt{2}\pi}{6M^2} \qquad D_1 = 0.61494625 - 2.55737/M^2$$
$$C_{-1} = \frac{\sqrt{2}}{\pi}\ln M \qquad D_3(0) = 0.8634168148\,. \tag{A7}$$

2. Two-dimensional square lattice

$$C_1 = 0.958091399 - 0.719/M^3 \qquad D_1 = 0.4826(1)$$
$$C_{-1} = 1.285764497 - 1.2415/M \qquad D_2 = 1.425(3) \tag{A8}$$
$$C_{-3} = 0.291348M + 0.1638 \qquad D_3(0) = 0.512870\,.$$

3. Two-dimensional triangular lattice

$$C_1 = 0.96774233 - 0.7802/M^3 \qquad D_1 = 0.4116(2)$$
$$C_{-1} = 1.23812403 - 1.15144/M \qquad D_2 = 1.224(2) \tag{A9}$$
$$C_{-3} = 0.237176M + 0.2384 \qquad D_3(0) = 0.3825896\,.$$

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